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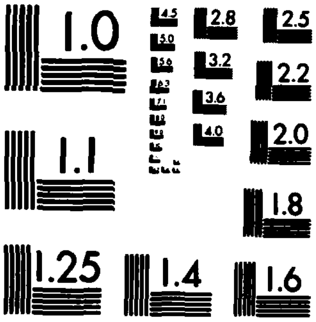

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# [illegible] and [illegible] Performance for a Class of Systems with Uncertain Parameters

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# FILTERING AND PREDICTION PERFORMANCE FOR A CLASS OF SYSTEMS WITH UNCERTAIN PARAMETERS

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# ABSTRACT

A covariance analysis technique using the Cramer-Rao lower bound for assessing filtering and prediction performance for a class of nonlinear systems is presented. The class of systems considered is nonlinear, deterministic, with unknown parameters. The validity of this technique for the problems considered is justified using local observability theory and unbiased estimation for nonlinear systems.

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# 1. INTRODUCTION

A covariance analysis technique by means of the Cramer-Rao lower bound for assessing filtering and prediction performance for a class of nonlinear systems is presented. The class of systems considered is nonlinear, deterministic, with unknown parameters. The unknown parameters can be constants or are known to follow nominal time functions (parametric or numerical). In the latter case, the proportional constant to the true time function is the unknown. In either cases, unknown parameters are to be jointly estimated with the state variables.

The above system definition fits very well to the problem of reentry trajectory estimation. The unknown parameter is the ballistic coefficient. In the high endoatmospheric region, the ballistic coefficient is a constant. In the lower altitude region, the ballistic coefficient is a time (altitude and Mach number) function. In certain applications, a nominal ballistic coefficient profile is known to the estimator.

The assumption that the nonlinear system is deterministic places certain restrictions on the generality of the ensueing analysis. In using an extended Kalman filter for state estimation in this case, a process noise variance is selected to represent the variability of the unknown parameter. On the other hand, if the maximum likelihood (batch) estimator is used with the assumption that a nominal parameter profile is available,

then the underlying system can be modeled as deterministic. For this latter situation, we therefore feel that the assumption of a deterministic system is valid for many applications.

The Cramer-Rao bound for nonlinear deterministic systems has been shown to be very tight for the trajectory estimation problem with angle-only measurements [5]. This is the basis of the analysis technique being introduced in this report. The problem of trajectory estimation with ballistic coefficient being the unknown parameter provided the motivation for the analysis method described herein.

This report is organized as follows. The problem considered in this report is defined using system and measurement equations in Section 2. The Cramer-Rao bound theory for deterministic nonlinear systems is reviewed in Section 3. The validity of the Cramer-Rao bound for the problems considered is justified using the local observability theory and the unbiased estimation. Covariance equations for filtering error and prediction errors are summarized in the fourth section.

## 2. PROBLEM DEFINITION

Consider the following continuous-time system with discrete measurement problem

$$\dot{\underline{x}} = \underline{f}\ (\underline{x},\ \underline{p}) \tag{2.1}$$

$$\underline{y}_k = \underline{h}\ (\underline{x}_k) + \underline{n}_k;\quad \underline{x}_k = \underline{x}(t_k) \tag{2.2}$$

where $\underline{x}$ and $\underline{y}$ are state and measurement vectors, $\underline{p}$ is the unknown parameter vector and $\underline{n}_k$ is the measurement noise vector which is a zero mean, white Gaussian sequence with covariance $R_k$.

Two cases for the parameter vector $\underline{p}$ are considered.

(1) $\underline{p}$ is an unknown constant vector.

(2) $\underline{p}$ follows a vector of profiles with known shape but uncertain in absolute value. We therefore have

$$p_i(t) = \alpha_i\ p_i^o(t)$$

where $p_i^o(t)$ is a time function (parametric or numerical) known to the estimator and denotes the i-th element of $\underline{p}^o(t)$. The corresponding i-th component of the true profile is denoted by $p_i(t)$ and the proportional constant $\alpha_i$ becomes the unknown constant to be estimated.

In either cases above, we augment the unknown constants with the state vector $\underline{x}$, the estimator is therefore to jointly estimate $\underline{x}$ and the unknown constants. Let $\underline{\gamma}$ denote either the constant vector $\underline{p}$ or the proportional constant vector $\underline{\alpha}$, we then have

$$\dot{\underline{\gamma}} = \underline{0} \tag{2.3}$$

and the augmented state vector $\underline{x}_a$ is

$$\underline{x}_a^T = [\underline{x}^T, \underline{\gamma}^T] \tag{2.4}$$

# 3. THE CRAMER-RAO BOUND FOR DETERMINISTIC NONLINEAR SYSTEMS

## 3.1 Review of the Cramer-Rao Bound Equations

The Cramer-Rao bound (CRB) on the covariance of estimating $\underline{x}(t_k)$ based upon measurement vectors $\underline{y}_o, \underline{y}_1, \ldots, \underline{y}_k$ for all unbiased estimators is given below.

$$I(\underline{x}_k) = \sum_i G_i^T H_i^T R_i^{-1} H_i G_i \tag{3.1}$$

$$P(\underline{x}_k) = I(\underline{x}_k)^{-1} \tag{3.1a}$$

where

$I(\underline{x}_k)$ = the Fisher's information matrix evaluated at $\underline{x}_k$.

$P(\underline{x}_k)$ = the Cramer-Rao bound evaluated at $\underline{x}_k$.

$G_i$ = $\Phi_i^{-1} G_{i+1}$; $i=k-1, k-2, \ldots, 1$.

$G_k$ = I (an identity matrix).

$\Phi_i$ = the solution of $\frac{\partial\, \Phi(t, \tau)}{\partial t} = F_t\, \Phi(t, \tau)$, $\Phi(\tau, r) = I$, for $\tau = t_i$, evaluated at $t = t_{i+1}$.

$F_t$ = the Jacobian matrix of $\underline{f}(\underline{x}(t))$.

$H_i$ = the Jacobian matrix of $h(\underline{x}_i)$.

$R_i$ = The measurement error covariance at time $t_i$.

The derivation of the above results may be found in [1], [2]. Notice that the $P(\underline{x}_k)$ may be re-evaluated at any time instance and this is accomplished via the appropriate choice of the composite transition matrix $G_i$.

In evaluating the performance of a certain nonlinear estimation problem, the Cramer-Rao bound provides several desirable features. It is easy to compute when compared against other bounds. When the maximum likelihood estimator (MLE) can be realized, the CRB and MLE go hand in hand. It is well-known that the MLE is consistent, asympotically efficient, and asymptotically Gaussian, [3]. The existence of MLE therefore guarantees that the CRB is at least asympotically achievable.

Clearly, the Cramer-Rao Bound (3.1a) does not exist if the Fisher's information matrix (3.1) is singular. This implies that there does not exist any unbiased estimator for $\underline{x}_k$ with finite estimation errors (variances). In the nonlinear systems theory, $I(\underline{x}_k)$ also ties with the nonlinear observability condition and the existence of a particular maximum likelihood estimation algorithm as we shall demonstrate next.

## 3.2 Local Observability Condition and Maximum Likelihood Estimation

In [4], the use of $I(\underline{x}_k)$ in examining the pointwise observability at $\underline{x}_k$ was presented.* It stated that if $I(\underline{x}_k)$ is positive definite, then the system is observable at $\underline{x}_k$. In here, we extend the above condition to a local sphere about $\underline{x}_k$. We first present the following theorem.

Theorem 3.1

Given a positive definite symmetric matrix A with eigenvalues $\lambda_n > \lambda_{n-1} \geq \ldots \geq \lambda_1 > 0$, if B is symmetric and $||A-B|| < \lambda_1$, then B is positive definite.

Proof:** Using the definition

$$\lambda_1 = \min_{||\underline{x}||=1} < A\underline{x}, \underline{x} >$$

and A-B being symmetric, we have

$$||A-B|| = \max_{||\underline{x}||=1} < (A-B)\, \underline{x}, \underline{x} >$$

Thus, for any unit vector $\underline{x}$,

---

* The observability Grammian of [4] is the same as $I(\underline{x}_k)$ with $R_i$ being set to an identity matrix.

** Proof of the theorem is due to Dr. R. B. Holmes.

$$
\begin{aligned}
< (A-B)\underline{x}, \underline{x} > &= < A\underline{x}, \underline{x} > - < B\underline{x}, \underline{x} > \\
&< \lambda_1 \\
&< < A\underline{x}, \underline{x} > ,
\end{aligned}
$$

Clearly, $<B\underline{x}, \underline{x}> > 0$, then B is positive definite. QED.

We can apply the above theorem to define a sphere around a point $\underline{x}$ where $I(\underline{x})$ is positive definite such that all points within this sphere will have observability Grammian being positive definite.

Theorem 3.2

If $I(\underline{y})$ is positive definite at $\underline{y}_o$, then one can construct a sphere $S = \{\underline{y} : ||\underline{y}-\underline{y}_o|| < r\}$ such that for all $\underline{y}$ in S, $I(\underline{y})$ is positive definite and the radius of S can be choosen as

$$r = \min \{r_1, r_2\}$$

where $||\underline{y}-\underline{y}_o|| < r_1$, $||I(\underline{y}) - I(\underline{y}_o)|| < L \, ||\underline{y}-\underline{y}_o||$, and $r_2=\lambda_1/L$.

Proof Given

$$||\underline{y}-\underline{y}_o|| < r_1$$

$$||I(\underline{y}) - I(\underline{y}_o)|| < L \, ||\underline{y}-\underline{y}_o|| < L \cdot r_1$$

Consider two cases:

(a) Assuming $r_1 < r_2$, or $L \cdot r_1 < \lambda_1$, then

$$||I(y) - I(y_o)|| < \lambda_1,$$

this gives $I(\underline{y}) > 0$.

(b) Assuming $r_2<r_1$, or $\lambda_1<L\cdot r_1$, then choose a new $\underline{y}$, $\underline{y}'$ such that

$$||\underline{y}' - \underline{y}_o|| < \lambda_1/L$$

Then $||I(\underline{y}')-I(\underline{y}_o)|| < L\cdot ||\underline{y}'-\underline{y}_o|| = \lambda_1$

this gives $I(\underline{y}') > 0$. QED.

The above theorem presents the fact that when $I(\underline{x}_k)$ is positive definite, then the reconstruction of $\underline{x}_k$ at a local region can be made and the size of this region (the sphere S) can be estimated. We next illustrate a particular realization of the Maximum Likelihood estimator where the existence of this algorithm also depends on the invertibility of $I(\underline{x}_k)$.

An iterative algorithm in implementing the Maximum Likelihood estimator for estimating the state vector with angle only measurements was presented in [5]. Let $\underline{x}_k^n$ denote the nth iteration on estimating $\underline{x}_k$, then the following algorithm, derived in [5], gives the n+1st iterative solution,

$$\underline{x}_k^{n+1} = \underline{x}_k^n + P(\underline{x}_k^n)\ [\sum_{i=1}^{k} G_i^n\ H_i^n\ R_i^{-1}(\underline{y}_i-\underline{h}\ (\underline{x}_i^n\ ))\ ] \qquad (3.2)$$

where the covariance matrix $P(\underline{x}_k^n)$ takes the same functional form as the Cramer-Rao Bound defined in (3.1). Clearly, the existence of $P(\underline{x}_k^n)$ is dependent upon the invertibility of $I(\underline{x}_k^n)$. It was shown in [5] that the above algorithm provides estimates which asymptotically approach the Cramer-Rao Bound. This was not surprising due to the theoretically justified property of the Maximum Likelihood estimator.

The derivation of (3.2) was based upon minimizing a quadratic error criterion, i.e.,

$$\min_{\underline{x}_k} \; J = \sum_{i=1}^{k} (\underline{y}_i - \underline{h}(\underline{x}_i))^T R_i^{-1} (\underline{y}_i - \underline{h}(\underline{x}_i)) \qquad (3.3)$$

In the case of designing a nonlinear observer, the Eq. (3.2) can be used simply be setting $R_i$ to an identity matrix. The convergence of (3.2) is guaranteed if the J above is convex in a local region about $\underline{x}_k$ and the initial guess $\underline{x}_k^o$ is contained in this region.

## 3.3 Summary

In this section, we have presented the fundamental equations on the Cramer-Rao bound for systems defined with (2.1) and (2.2). We have also tied the relationship between the invertibility of the Fisher's information matrix to the local

observability of nonlinear systems and the existence of a particular Maximum Likelihood algorithm. we summarize our findings below.

(1) The computation of the Cramer-Rao bound is possible only if the Fisher's information matrix is positive definite.

(2) When this is the case, the nonlinear system is locally observable.

(3) A maximum likelihood estimation algorithm, which also requires the Fisher's information matrix to be nonsingular, can be constructed. In the noise free measurement case, this algorithm becomes an observer and the convergence is guaranteed if the quadratic error criterion is convex about $\underline{x}_k$ and the initial guess $\underline{x}_k^o$ is contained in this region.

## 4. ERROR COVARIANCE EQUATIONS

With the use of the Cramer-Rao Bound for estimation performance evaluation justified, we now present computational algorithms for filtering and prediction errors.

### 4.1 Filtering Errors

The computation of (3.1) involves matrix inversion. When sufficient number of measurements have been collected so that the information matrix $I(\underline{x}_k)$ becomes nonsingular, then Eq. (3.1) can be replaced by the familiar matrix Riccati equation. Examining (3.1), one can write

$$I(\underline{x}_{k+1}) = \sum_{i=1}^{k+1} G_i^T H_i^T R_i^{-1} H_i G_i$$

$$= H_{k+1}^T R_{k+1}^{-1} H_{k+1} + \Phi_k^{-T} I(\underline{x}_k) \Phi_k^{-1} \qquad (4.1)$$

Let

$$\tilde{P}_{k+1} = \Phi_k P(\underline{x}_k) \Phi_k^T \qquad (4.2)$$

One can obtain the following result with direct application of the Matrix Inversion Lemma to Eq. (4.1).

$$P(\underline{x}_{k+1}) = \tilde{P}_{k+1} [I- H_{k+1}^T(H_{k+1}^T\tilde{P}_{k+1}H_{k+1}+ R_{k+1})^{-1}H_{k+1}\tilde{P}_{k+1}] \qquad (4.3)$$

The computation of $\tilde{P}_{k+1}$, Eq. (4.2), can also be replaced by solving for $\tilde{P}_{k+1}$ using the following matrix differential equation,

$$\dot{P} = F_t P + P F_t^T , \qquad t \in [t_k, t_{k+1}] \qquad (4.4)$$

with initial condition $P(\underline{x}_k)$ at $t = t_k$.

Equations (4.3) and (4.4) give the filtering error. They are applied as soon as $I(\underline{x}_k)$ becomes nonsingular.

## 4.2 Prediction Errors

The prediction error equation is (4.4) (or (4.2)), by solving it at the time desired with the initial condition set at the last point of measurements.

In the problem of trajectory estimation with the ballistic coefficient as the unknown parameter, two situations may occur for the problem of trajectory prediction. This is due to the fact that the variations of ballistic coefficients (i.e., deviations from a nominal time profile) are usually known to within a priori bounds. When the ballistic coefficient estimation (filtering) error is below those errors characterized by a priori bounds, the trajectory prediction error is obtained by solving Eq. (4.4) (or (4.2)). On the other hand when such is not the case, the trajectory prediction error should only be limited to errors induced by the a priori bounds. In this latter case, the error equation can be derived using the following linear equation approximation. Let the linear system

be modeled as

$$\dot{\underline{x}} = F\ \underline{x} + G\ \underline{\gamma} \tag{4.5}$$

Notice that the unknown parameter is treated as a driving force term. The solution of (4.5) can be obtained via the transition matrix $\Phi(t, \tau)$ as

$$x_t = \Phi(t, t_o)\ \underline{x}_o + \int_{t_o}^{t} \Phi(t, \tau)\ G\ \underline{\gamma}(\tau)\ d\tau \tag{4.6}$$

where $\underline{x}_o$ is the initial condition. Let $\tilde{\underline{x}}_t$ denoted a perturbed solution due to perturbation $\delta\underline{\gamma}$ in the parameter vector $\underline{\gamma}$, one obtains

$$\tilde{\underline{x}}_t = \Phi\ (t, t_o)\ \underline{x}_o + \int_{to}^{t} \Phi(t, \tau)\ G(\delta+\delta\underline{\gamma})\ d\tau \tag{4.7}$$

Let $\sum$ denote the outer product of the trajectory perturbation $\delta\underline{x}_t$, it can be shown that

$$\delta\underline{x}_t = \int_{t_o}^{t} \Phi(t, \tau)\ G\delta\underline{\gamma}\ d\tau \tag{4.8}$$

and

$$\sum = \delta\underline{x}_t\delta\underline{x}_t^T \tag{4.9}$$

The above analysis suggests the following two separate procedures for trajectory prediction when a priori bounds on the unknown parameter are available.

(1) When the parameter estimation error is smaller than the a priori bound, use the covariance differential equation (Eq. (4.2) or (4.4)) to solve for trajectory prediction error.

(2) When the parameter estimation error is large, calculate estimation error assuming perfect knowledge on the parameter value, then obtain trajectory prediction error using trajectory perturbation Eqs. (4.8) - (4.9).

## 5. SUMMARY

In this report, we have presented equations for calculating filtering and prediction errors for systems with uncertain parameters. The results are based upon the Cramer-Rao bound on covariance in state estimation. This analysis is motivated by the problem of trajectory estimation with uncertain parameters.

The major findings of this report are summarized below.

(1) Unknown constant parameters are modeled as constant state variables.

(2) Unknown time-varying parameters with known time profiles require the modeling of proportional constants as constant state variables.

(3) The use of the Cramer-Rao Bound as an analysis tool is discussed and its relationship with the nonlinear observability condition and the existence of a maximum likelihood algorithm is explored.

(4) Filtering errors are obtained by solving the Cramer-Rao bound equation.

(5) Prediction errors due to filtering error alone or with the knowledge of parameter a priori bounds are also obtained.

Application of the analysis discussed herein to the trajectory estimation problem will be published in a future report.

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A covariance analysis technique using the Cramer-Rao lower bound for assessing filtering and prediction performance for a class of nonlinear systems is presented. The class of systems considered is nonlinear, deterministic, with unknown parameters. The validity of this technique for the problems considered is justified using local observability theory and unbiased estimation for nonlinear systems.

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